

NUMBERS AND FUNCTIONS IN QUANTUM FIELD THEORY

OLIVER SCHNETZ

ABSTRACT. We review recent results in the theory of numbers and single-valued functions on the complex plane which arise in quantum field theory.

1. INTRODUCTION

Quantum field theories (QFTs) are fundamental theories of physical interactions. Physical QFTs are the Electroweak theory which combines electromagnetism with the weak interaction, Quantumchromodynamics which describes the interaction between quarks and gluons, and ϕ^4 theory for the Higgs boson. Gravity has not yet found a QFT formulation.

Although QFTs are experimentally very well confirmed (see e.g. the anomalous magnetic moment of the electron for an impressing example [2, 27, 32]), a complete mathematical understanding of QFTs is lacking. On the one hand there are fundamental questions like the existence and structure of QFTs. On the other hand there is demand for practical tools to perform QFT calculations. Due to the mathematical difficulty of QFTs progress is modest. Here, we report on some recent results in both directions. For a while it seemed possible that the number content of QFT is given by multiple zeta values (MZVs) which are multiple sums that generalize the Riemann zeta function at positive integer arguments. Assuming standard conjectures it has now been proved that this is not the case [14, 13].

With the theory of graphical functions, a tool was developed to perform multiloop calculations in massless scalar field theories [42, 25]. The first notable breakthrough was the proof of the zig-zag conjecture [5, 15] which gives an explicit formula for periods of zig-zag graphs (see Thm. 4.8). For more general applications it was necessary to introduce a novel family of single-valued functions on the complex plane: generalized single-valued hyperlogarithms (GSVHs). It was possible to extend all essential results from single-valued multiple polylogarithms to GSVHs [44]. A MapleTM package was developed that made it possible to calculate many periods in ϕ^4 theory up to 11 loops [46]. With a large amount of data available, it was possible to connect the structure of ϕ^4 periods to the Galois theory of algebraic integrals [11, 12, 37].

To make further contact to physics it is necessary to regularize integrals which diverge in four dimensions. This is often done by generalizing to $4 - \epsilon$ dimensions (which can be defined in a parametric representation of QFT integrals [28]). Using GSVHs it was possible to obtain ϵ -expansions for QFT periods and graphical functions. With a fully automated computer program the QFT β -function and the self energy was calculated to six loops, and the γ -function to seven loops in the minimal subtraction scheme. The seven loop calculation of the beta function is currently running.

Habilitationsschrift.

ACKNOWLEDGEMENTS

I am very grateful to D. Kreimer and F. Knop for their support. I am also deeply indebted to my co-authors F. Brown, E. Panzer, and K. Yeats. Most of the results in this report were found when the author was visiting scientist at the Humboldt University, Berlin. The author is supported by DFG grant SCHN 1240/2-1.

2. GENERAL IDEA

In QFT graphs are used to symbolize integrals. We first have to fix a (space-time) dimension d which, for brevity, we connect to the parameter λ according to

$$d = 2 + 2\lambda > 2.$$

The graphs allowed depend on the QFT chosen. Here, we are mainly interested in ϕ^4 theory which limits the vertex degree to four. Feynman rules translate graphs to integrals. They exist in momentum and in position space with integration over d -dimensional variables associated to independent cycles or vertices, respectively. For massive theories one needs to use momentum space to obtain explicitly algebraic integrands. Here, we are interested in massless theories which allows us to use position space.

The general setup is as follows. Assume G is a graph with edges $\mathcal{E}(G)$ and vertices $\mathcal{V}(G)$. We do not assume that every vertex in G has maximum degree four. Every edge $e \in \mathcal{E}(G)$ has a weight $\nu_e \in \mathbb{R}$. We assume that G has no self loops (tadpoles). This is common in dimensionally regularized massless theories. The edge weight is additive, i.e. a multiple edge in G is equivalent to a single edge with weight equal to the weight sum of the multiple edge. Therefore, we only need to consider single edges. We split the set of vertices into ‘internal’ vertices $\mathcal{V}^{\text{int}}(G)$ and ‘external’ vertices $\mathcal{V}^{\text{ext}}(G)$. To every vertex we associate a d dimensional variable and do not distinguish between the vertex and the variable. We use $x_i, i = 1, \dots, V^{\text{int}}(G) = |\mathcal{V}^{\text{int}}(G)|$ for internal vertices and $z_i, i = 1, \dots, V^{\text{ext}}(G) = |\mathcal{V}^{\text{ext}}(G)|$ for external vertices. To every edge $e = \{u, v\} \in \mathcal{E}(G)$ between the two vertices $u, v \in \mathcal{V}(G)$ (internal or external) we associate a quadric Q_e which is given by the Euclidean distance between (the variables associated to) u and v ,

$$Q_e(u, v) = \|u - v\|^2 = (u_1 - v_1)^2 + \dots + (u_d - v_d)^2.$$

Assume the graph G has the property that the following integral exists

$$(1) \quad f_G^{(\lambda)}(z_1, \dots, z_{V^{\text{ext}}}) = \left(\prod_{v=1}^{V^{\text{int}}(G)} \int_{\mathbb{R}^d} \frac{d^d x_v}{\pi^{d/2}} \right) \frac{1}{\prod_{e \in \mathcal{E}(G)} Q_e^{\lambda \nu_e}}.$$

Due to translational and scale invariance the integral can only exist if G has at least two external vertices. In the case of exactly two external vertices the integral is determined up to a constant by these symmetries ($E(G) = |\mathcal{E}(G)|$)

$$(2) \quad f_G^{(\lambda)}(z_1, z_2) = P(G) \|z_1 - z_2\|^{dV^{\text{int}} - 2\lambda E(G)}.$$

The number $P(G) \in \mathbb{R}_+$ is the Feynman period of G . Without loss of information we may set $z_1 = '0' = (0, \dots, 0)$ and $z_2 = '1' = (1, 0, \dots, 0)$ (we may associate to the vertex 1 any unit vector in \mathbb{R}^d) and obtain

$$f_G^{(\lambda)}(0, 1) = P(G).$$

In the case of three external vertices we can again exploit the symmetry of the integral to reduce the number of variables. In this case we may use a complex variable z (and

its complex conjugate \bar{z}) to describe the functional behavior of $f_G^{(\lambda)}$. We obtain the ‘graphical function’ [42]

$$f_G^{(\lambda)}(z) = f_G^{(\lambda)}\left(0, 1, \left(\frac{z + \bar{z}}{2}, \frac{z - i\bar{z}}{2i}, 0, \dots, 0\right)\right).$$

In full generality $f_G^{(\lambda)}(z)$ is a single-valued real analytic function on $\mathbb{C} \setminus \{0, 1\}$ [25] with the residual symmetry

$$f_G^{(\lambda)}(z) = f_G^{(\lambda)}(\bar{z}).$$

The benefit of complex numbers is that the quadrics between external vertices factorize

$$Q_{\{0,1\}} = 1, \quad Q_{\{0,z\}} = z\bar{z}, \quad Q_{\{1,z\}} = (z-1)(\bar{z}-1).$$

General graphs with four or more vertices lead to functions which depend on a variable in \mathbb{R}^3 which does not have this factorizing property. An exception are ‘conformal’ graphs with four external vertices where every internal vertex has degree $2d/(d-2)$. In this case one may use an inversion $x_i \mapsto x_i/||x_i||^2$ to reduce the integral to the case of three external vertices e.g. [23], [39]. Here, we go the opposite direction and ‘complete’ graphical functions to conformal graphs with four external vertices [42]. We will see in Section 4.1 that this is useful to exploit the full symmetry of graphical functions.

In the following we restrict ourselves to the above two cases, periods and graphical functions. In ϕ^4 theory four point functions are formally conformal. However, only the tree level contribution is convergent. A practical tool to resolve divergences is to transform all integrals in a parametric form using the Schwinger or Feynman trick [28]. In parametric form one integrates over one-dimensional variables associated to the edges of the graph. The dimension d enters the integrand as an exponent. So, it is possible to consider d as a parameter and use analytic continuation to $d = 4 - \epsilon$ (losing conformal invariance). All parametric integrals have Laurant expansions at $\epsilon = 0$. Graphical functions are amenable to such a procedure. A general parametric representation of graphical functions is given in [25] (which generalizes a formula in [33]).

3. NUMBERS

A graph G is called ϕ^4 if G has maximum vertex degree four. The period of a ϕ^4 graph in $d = 4$ dimensions is a ϕ^4 period. The loop order of G is the number of independent cycles in G .

3.1. Completion. We ‘complete’ a graph G with two external vertices 0 and 1 by adding a new external vertex which we give the label ‘ ∞ ’ (as reference to conformal symmetry) [40]. We add edges from ∞ to all internal vertices in G such that every internal vertex has (conformal) degree $2d/(d-2)$. Finally, we add a weighted triangle with vertices 0, 1, ∞ such that the completed graph becomes $2d/(d-2)$ regular. We denote the completion of G by \bar{G} . It is easy to see that completion is always possible and unique. If we employ the Feynman rule that every edge e adjacent to ∞ has quadric $Q_e = 1$ we find that the period (1) of the completed graph equals the period of the original graph.

The power of completion is that the period of a completed graph does not depend on the choice of the external vertices 0, 1, ∞ . This was proved in four dimensions in Theorem and Definition 2.7 in [40]. The d dimensional case is strictly analogous. Hence, in \bar{G} we do not need the distinction between internal and external vertices. We henceforth consider \bar{G} as an unlabeled graph.

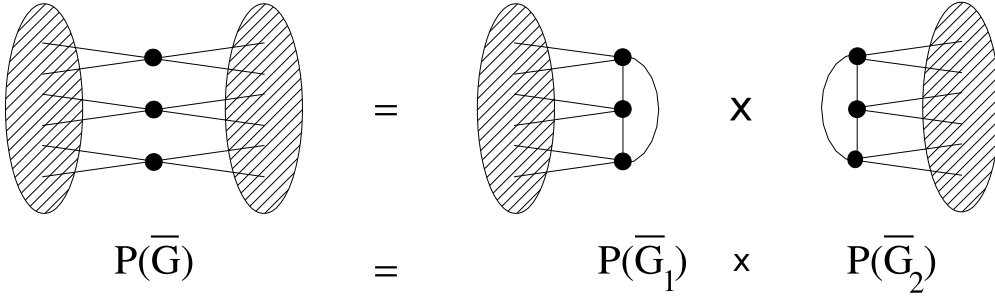


FIGURE 1. Vertex connectivity 3 leads to a product of periods.

Obviously different graphs G_1 and G_2 can have the same completion. In this case completion implies equality of their periods: $P(G_1) = P(G_2)$ if $\overline{G}_1 = \overline{G}_2$. This identity on periods was already used in [5]. In this sense completion is a tool to organize equivalence classes of graphs with identical period. The loop order of a completed graph \overline{G} is defined as the number of independent cycles in the uncompleted graph G . So, by definition, completion does not change the loop order.

3.2. Existence. In four dimensions the period of a completed graph \overline{G} with edge-weights 1 exists if and only if \overline{G} is internally 6-connected. This means that the only way to cut \overline{G} with less than 6 edge cuts is to separate off a vertex (Prop. 2.6 in [40]).

In general, a Feynman period $P(G)$ can be considered as a graphical function (see Sect. 4.1) with an isolated external vertex z . Existence of $P(G)$ is hence a special case of the criterion for the existence of graphical functions in Thm. 4.3.

A completed graph with existing period is called completed primitive [40].

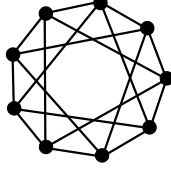
3.3. Product identity. A completed primitive graph \overline{G} has vertex connectivity ≥ 3 (the vertex connectivity is the minimum number of vertices which, when removed, split the graph). The period of a completed primitive graph \overline{G} with vertex-connectivity three factorizes in the way depicted in Figure 1. Reversely, completed primitive graphs \overline{G}_1 , \overline{G}_2 with triangles can be glued along triangles to provide a completed primitive graph with period $P(\overline{G}_1)P(\overline{G}_2)$. The case $d = 4$ with unit edge-weights was treated in Thm. 2.10 in [40]. The general case is analogous.

A completed primitive graph with vertex connectivity three is called reducible in [40], otherwise it is irreducible. A complete list of all irreducible completed primitive graphs up to eight loops (in four dimensions with unit edge weights) is given in Table 3 at the end of this report.

Because not all completed primitive graphs have triangles (see e.g. $P_{6,4}$ in Table 3) it is unclear if, in general, the product of Feynman periods is a Feynman period. In particular, we may ask if the \mathbb{Z} -span of ϕ^4 periods is a ring. Or, weaker, if the \mathbb{Q} -span of ϕ^4 periods is an algebra.

3.4. Twist and Fourier identity. There are two more identities on ϕ^4 periods known. The (for graphs with many vertices) frequent twist identity and the rare Fourier identity [40]. The Fourier identity was already used in [5]. The first example of a twist identity which is not also explained by the Fourier identity appears at eight loops. Twist and Fourier identities are listed in Table 3.

It is unclear if completion, twist, and Fourier identities are sufficient to explain all identities between ϕ^4 periods.

FIGURE 2. The completed graph $P_{7,11}$.

ℓ	wt	number	value
1	0	$Q_0 = 1$	1
3	3	$Q_3 = \zeta(3)$	1.202 056 903 159
4	5	$Q_5 = \zeta(5)$	1.036 927 755 143
5	7	$Q_7 = \zeta(7)$	1.008 349 277 381
6	8	$Q_8 = N_{3,5}$	0.070 183 206 556
	9	$Q_9 = \zeta(9)$	1.002 008 392 826
7	10	$Q_{10} = N_{3,7}$	0.090 897 338 299
	11	$Q_{11,1} = \zeta(11)$	1.000 494 188 604
		$Q_{11,2} = -\zeta(3, 5, 3) + \zeta(3)\zeta(5, 3)$	0.042 696 696 025
		$Q_{11,3} = P_{7,11}$, Eq. (3)	200.357 566 429
8	12	$Q_{12,1} = N_{3,9}$	0.096 506 102 637
		$Q_{12,2} = N_{5,7}$	0.020 460 547 937
		$Q_{12,3} = \pi^{12}/10!$	0.254 703 808 841
	13	$Q_{13,1} = \zeta(13)$	1.000 122 713 347
		$Q_{13,2} = -\zeta(5, 3, 5) + 11\zeta(5)\zeta(5, 3) + 5\zeta(5)\zeta(8)$	5.635 097 688 692
		$Q_{13,3} = -\zeta(3, 7, 3) + \zeta(3)\zeta(7, 3) + 12\zeta(5)\zeta(5, 3) + 6\zeta(5)\zeta(8)$	6.725 631 947 085
		$Q_{13,4} = P_{8,33}$ [46]	468.038 498 992

Table 1: List of ϕ^4 -transcendentals up to loop order eight. The list is incomplete at loop order eight.

ℓ	wt	base
6	8	$N_{3,5} = \frac{27}{80}\zeta(5, 3) + \frac{45}{64}\zeta(5)\zeta(3) - \frac{261}{320}\zeta(8)$
7	10	$N_{3,7} = \frac{423}{3584}\zeta(7, 3) + \frac{189}{256}\zeta(7)\zeta(3) + \frac{639}{3584}\zeta(5)^2 - \frac{7137}{7168}\zeta(10)$
8	12	$N_{3,9} = \frac{27}{512}\zeta(4, 4, 2, 2) + \frac{55}{1024}\zeta(9, 3) + \frac{231}{256}\zeta(9)\zeta(3) + \frac{447}{256}\zeta(7)\zeta(5) - \frac{9}{512}\zeta(3)^4$ $- \frac{27}{448}\zeta(7, 3)\zeta(2) - \frac{189}{128}\zeta(7)\zeta(3)\zeta(2) - \frac{1269}{1792}\zeta(5)^2\zeta(2) + \frac{189}{512}\zeta(5, 3)\zeta(4)$ $+ \frac{945}{512}\zeta(5)\zeta(3)\zeta(4) + \frac{9}{64}\zeta(3)^2\zeta(6) - \frac{7322453}{5660672}\zeta(12)$ $N_{5,7} = -\frac{81}{512}\zeta(4, 4, 2, 2) + \frac{19}{1024}\zeta(9, 3) - \frac{477}{1024}\zeta(9)\zeta(3) - \frac{4449}{1024}\zeta(7)\zeta(5) + \frac{27}{512}\zeta(3)^4$ $+ \frac{81}{448}\zeta(7, 3)\zeta(2) + \frac{567}{128}\zeta(7)\zeta(3)\zeta(2) + \frac{3807}{1792}\zeta(5)^2\zeta(2) - \frac{567}{512}\zeta(5, 3)\zeta(4)$ $- \frac{2835}{512}\zeta(5)\zeta(3)\zeta(4) - \frac{27}{64}\zeta(3)^2\zeta(6) + \frac{3155095}{5660672}\zeta(12)$

Table 2: Conversion of the $N_{a,b}$ s in Table 1 into MZVs.

3.5. The number content of ϕ^4 periods. Up to five loops there exists at most one ϕ^4 period per loop order. These periods are the first instances of the infinite family

of zig-zag periods (see Figure 5). The periods of the zig-zag family are known as a rational multiple of the Riemann zeta function at odd arguments (see Thm. 4.8).

At six loops the ϕ^4 periods $P_{6,3}$ and $P_{6,4}$ have a zeta double sum of weight eight. In general, Feynman periods are often multiple zeta values (MZVs) which are \mathbb{Q} linear combinations of multiple zeta sums

$$\zeta(n_d, \dots, n_1) = \sum_{k_d > \dots > k_1 \geq 1} \frac{1}{k_d^{n_d} \dots k_1^{n_1}} \quad \text{with } n_i \in \mathbb{Z}_{>0}, n_d \geq 2.$$

At seven loops there exists a single period, $P_{7,11}$, which is not expressible in terms of MZVs. The period $P_{7,11}$ (see Figure 2) features an extension of MZVs by sixth (or third) roots of unity. The result is best given in the f^6 -alphabet with respect to the even parity Deligne basis [37]:

$$\begin{aligned} \frac{P_{7,11}}{i\sqrt{3}} &= -\frac{332\,262}{43} f_8^6 f_3^6 + \frac{54\,918}{55} f_6^6 f_5^6 + \frac{1\,134}{13} f_4^6 f_7^6 - \frac{1\,874\,502}{3\,485} f_2^6 f_9^6 \\ (3) \quad &- 5\,670 f_2^6 f_3^6 f_3^6 f_3^6 - \frac{3\,216\,912\,825\,399\,005\,402\,331\,281\,812\,377\,062\,149}{10\,264\,478\,246\,467\,100\,965\,990\,650\,592\,350\,882\,000} (\pi i)^{11}. \end{aligned}$$

There exists a lengthy conversion of the period $P_{7,11}$ in terms of multiple polylogarithms evaluated at primitive sixth roots of unity. The period $P_{7,11}$ was calculated by Erik Panzer (using his program `HyperInt`) in his PhD thesis [34] in terms multiple polylogarithms. The conversion into the f -alphabet was achieved by the Maple package `hyperlog_procedures` [46].

At eight loops still most periods are MZVs. Beyond MZVs we found for the period $P_{8,33}$ an expression with weight 13 which is similar to $P_{7,11}$. Moreover, there exist four periods of an entirely new type. The geometry underlying these periods is no longer a punctured sphere $\mathbb{C} \setminus \{0, 1, \dots\}$. Instead of point punctures we obtain in two cases K3 surfaces embedded in a higher dimensional space [14]. In the other two cases we found a threefold and a fivefold, respectively. All four varieties are modular of low level [16].

Beyond eight loops periods associated to non modular varieties are expected [16].

3.6. The coaction conjectures. In (3) a letter f^6 of even lower weight (subscript) appears only on the leftmost position. This is a consequence of a Galois structure in ϕ^4 periods.

In [31] M. Kontsevich and D. Zagier defined the \mathbb{Q} algebra of periods \mathcal{P} as integrals of rational forms over \mathbb{Q} . Feynman periods are periods in this sense. By general philosophy [1], there should exist a Galois coaction [26, 10, 12] on \mathcal{P} ,

$$(4) \quad \Delta: \mathcal{P} \longrightarrow \mathcal{P}^{\text{dr}} \otimes_{\mathbb{Q}} \mathcal{P},$$

where the left hand side of the tensor product is the Hopf algebra of deRahm periods. In the special case of ϕ^4 periods the right hand side seems to be severely restricted. A mathematical theory with first results is in [11]. The data of approximately 300 known ϕ^4 periods up to 11 loops led to the following possible scenarios [37] for the \mathbb{Q} algebra \mathcal{P}_{ϕ^4} generated by ϕ^4 periods.

Scenario 1.

$$(5) \quad \Delta: \mathcal{P}_{\phi^4} \longrightarrow \mathcal{P}^{\text{dr}} \otimes_{\mathbb{Q}} \mathcal{P}_{\phi^4}.$$

Scenario 2.

$$(6) \quad \Delta: \mathcal{P}_{\phi^4, \leq n} \longrightarrow \mathcal{P}^{\text{dr}} \otimes_{\mathbb{Q}} \mathcal{P}_{F, \leq n-1},$$

I.e. for a given ϕ^4 period of n loops the contributions on the right hand side of the tensor product is in the \mathbb{Q} vector space spanned by Feynman periods of all graphs with at most $n - 1$ loops.

Scenario 3. Scenario 1 becomes true only if one includes ϕ^4 periods of graphs with subdivergences which are renormalized in suitable way.

Scenario 4. None of the Scenarios 1, 2, 3 is true.

Note that \mathcal{P}_{ϕ^4} and \mathcal{P}_F are very sparse in \mathcal{P} , so that the Scenarios 1 and 2 have huge predictive power on \mathcal{P}_{ϕ^4} . Presumably \mathcal{P}_{ϕ^4} also is very sparse in \mathcal{P}_F . The coaction conjectures are (5) and (6).

3.7. The c_2 invariant. The c_2 invariant assigns to every graph with at least three vertices an infinite sequence which is indexed by prime powers $q = p^n$ [41],

$$c_2 : G \mapsto (c_2(G)_q)_q = (c_2(G)_2, c_2(G)_3, c_2(G)_4, c_2(G)_5, c_2(G)_7, \dots).$$

For fixed q , $c_2(G)_q$ has a value in $\mathbb{Z}/q\mathbb{Z}$. The c_2 invariant is linked to the period integral in four dimensions. With a metric signature $(+, -, +, -)$ instead of the Euclidean signature it can be defined via the point-count N_q of the singular locus of the period integral (1) over the finite field \mathbb{F}_q [17, 22]. For graphs with at least three vertices, N_q is divisible by q^2 and we define

$$c_2(G)_q \equiv N_q/q^2 \pmod{q}.$$

In practice, it is more efficient to calculate the c_2 invariant in parametric space where the above equation is still valid [41], [14].

The power of the c_2 invariant is twofold: Firstly, there exist powerful tools which make it possible to determine the c_2 invariant for many graphs. If the c_2 invariant cannot be fully calculated, it is still possible to determine the c_2 invariant for small primes [16, 48]. For all completed primitive ϕ^4 graphs up to ten loops the c_2 is known for at least the first six primes (in most cases much more) [16].

Secondly, the c_2 invariant has some predictive power on the period. In particular, if two graphs have the same period, they are conjectured to have the same c_2 invariant,

$$P(G_1) = P(G_2) \Rightarrow c_2(G_1) = c_2(G_2).$$

All graphs with c_2 invariant -1 (i.e. $c_2(G)_q \equiv -1 \pmod{q}$ for all q) should have an MZV period. If the c_2 invariant is $-z_2$, with

$$(7) \quad z_N(q) = \begin{cases} 1 & \text{if } N|q-1, \\ 0 & \text{if } \gcd(N, q) > 1, \\ -1 & \text{otherwise,} \end{cases}$$

then the period is expected to be an Euler sum (due to the coaction conjectures in many cases these periods are still MZVs [37]). The c_2 invariant of the periods $P_{7,11}$ and $P_{8,33}$ are $-z_3$. This links the c_2 invariant to the sixth roots of unity which exist in the period (3). The connection between ϕ^4 periods and higher dimensional geometries (in some graphs of at least eight loops) is proved with the c_2 invariant [14, 16].

The c_2 invariant was also used to prove (assuming standard transcendental conjectures) that not all ϕ^4 periods are MZVs or extensions of MZVs by algebraic numbers [14], [13]. Concretely, it was shown that $P_{8,37}$ is linked in such a way to the geometry of a K3 surface (which is modular of weight 3 level 7) that the ‘motivic’ period cannot be mixed Tate.

3.8. The Hepp invariant. For any graph G with edge weights $\{\nu_e\}$, $e \in \mathcal{E}(G)$ and $a \in \mathbb{C}$ we recursively define the following Hepp invariant:

$$(8) \quad H_a(G) = \frac{\sum_{e \in \mathcal{E}(G)} \nu_e H_a(G \setminus e)}{N_G - ah_1(G)},$$

where $N(G) = \sum_{e \in \mathcal{E}(G)} \nu_e$ is the sum of edge weights and $h_1(G)$ is the (Betti) number of independent cycles in G . A graph with no edges has Hepp invariant 1.

Lemma 3.1. *The Hepp invariant has the following properties:*

- (1) *If the removal of the edge e disconnects the graph G then $H_a(G) = H_a(G \setminus e)$.*
- (2) *If G has vertex connectivity ≤ 1 then H_a factorizes, i.e. $H_a(G)$ is the product of the Hepp invariants of its components (cutting at split vertices without removing edges).*
- (3) *Assume a vertex v in G is adjacent to exactly two edges e and f with weights ν_e and ν_f . We construct a smaller graph G' by contracting the edge e (or f) in G and giving f (e) the weight $\nu_e + \nu_f$. Then $H_a(G) = H_a(G')$.*

Proof. Straight forward induction over the number of edges in G . \square

The above lemma can be used to efficiently calculate the Hepp invariant of reasonably large graphs. Note that every forest has Hepp invariant 1.

If a graph G has a Feynman period in $d = 4$ dimensions then $H_a(G)$ has a simple pole at $a = 2$. We define the Hepp period of G as the residue,

$$(9) \quad H(G) = -h_1(G) 2^{-h_1(G)} \text{res}_{a=2} H_a(G).$$

Erik Panzer recently found that the Hepp period is closely related to the Feynman period [36]. Firstly, it approximates the Feynman period surprisingly well,

$$(10) \quad P(G) \approx 0.545^{h_1(G)-1} H(G)^{1.355}.$$

with an error of a few percent. The numerical estimates for the unknown eight loop periods in Table 3 were obtained by a refinement of (10).

Even more surprisingly, the Hepp period seems to know all identities between periods.

Conjecture 3.2 (E. Panzer).

$$(11) \quad P(G_1) = P(G_2) \Leftrightarrow H(G_1) = H(G_2)$$

At eight loops the above conjecture requires $P_{8,30} = P_{8,36}$ and $P_{8,31} = P_{8,35}$. There is no sequence of twists or Fourier identities known that proves these identities.

Assuming Conj. 3.2 for completion, the product identity (Sect. 3.3) was proved by E. Panzer and independently by K. Yeats to hold for Hepp periods [36], [49].

It is easy to see that $P(G) \leq 2^{h_1(G)} H(G)$ which is the classical Hepp bound for periods. Erik Panzer has achieved a certain refinement of this bound [36]. We conjecture that the Hepp period is a (crude) upper bound for the period, $P(G) \leq H(G)$.

4. FUNCTIONS

Let G be a graph with external vertices $0, 1, z$ such that the graphical function $f_G^{(\lambda)}$ exists. This version can be utilized to define graphical functions in complex dimensions which is necessary in the context of dimensional regularization.

4.1. Completion. Like in the case of periods, completion exploits conformal symmetry to handle equivalence classes of closely related graphical functions. Again, we add an external vertex ' ∞ ' which connects to all internal vertices of G with weights that give the internal vertices weighted degree $2d/(d-2)$. Now, we add edges $\{z, \infty\}$, $\{0, 1\}$, $\{0, \infty\}$, $\{1, \infty\}$ such that all external vertices have weighted degree 0. This provides the completion \overline{G} of G . The graphical function $f_{\overline{G}}^{(\lambda)}$ of the completed graph \overline{G} is defined as the graphical function of $\overline{G} \setminus \infty$ (edges adjacent to ∞ have quadric 1). Clearly, the graphical function does not change under completion. Completion is always possible

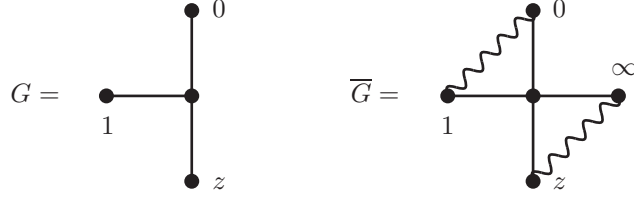


FIGURE 3. A four dimensional graphical function with three edges and its completion. The wiggly lines refer to edge weights -1 .

and unique. In three or four dimensions we also know that the completed graph \overline{G} has integer edge weights if G has (Lemma 3.18 in [42]).

A permutation of external vertices in a completed graph results in a Möbius transform of the argument z :

Theorem 4.1 (Theorem 3.20 in [42]). *Let $\sigma : \{0, 1, z, \infty\} \rightarrow \{0, 1, \phi(z), \infty\}$ be a permutation of $\{0, 1, z, \infty\}$ followed by a Möbius transformation $z \mapsto \phi(z)$ such that σ preserves the cross ratio $(0, 1; z, \infty)$, i.e.*

$$\frac{-z}{1-z} = \frac{(\sigma(0) - \sigma(z))(\sigma(1) - \sigma(\infty))}{(\sigma(1) - \sigma(z))(\sigma(0) - \sigma(\infty))}.$$

Then $f_{\overline{G}}^{(\lambda)} = f_{\sigma(\overline{G})}^{(\lambda)}$, where the Möbius transformation in the label of $\sigma(\overline{G})$ acts on the argument of the graphical function. In particular, $f_{\overline{G}}^{(\lambda)}$ is invariant under double transpositions of external labels.

Because edges between external vertices produce trivial factors, completed graphical functions with stripped off edges between external vertices are equivalence classes graphical functions of the same type. In four dimensions all those graphical functions with edge weights one and at most seven vertices are known [46].

Example 4.2. The smallest non-trivial graphical function has four vertices and three edges (see Figure 3). In four dimensions, this graphical function is given by the Bloch-Wigner dilogarithm (see e.g. [50]).

$$f_G^{(\lambda)}(z) = \frac{4iD(z)}{z - \overline{z}},$$

with

$$(12) \quad D(z) = i \operatorname{Im}(\operatorname{Li}_2(z) + \ln(1-z) \ln|z|).$$

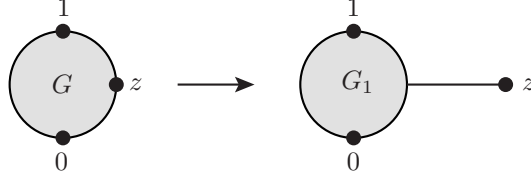
Theorem 4.1 and $f_G^{(\lambda)}(z) = f_G^{(\lambda)}(\overline{z})$ reflect the symmetries of D .

4.2. Existence. Existence of the graphical function $f_G^{(\lambda)}$ is best formulated in terms of the completion \overline{G} .

Theorem 4.3 (Lemma 3.19 in [42]). *The graphical function $f_{\overline{G}}^{(\lambda)}$ exists if and only if for any vertex subset $V \subset \mathcal{V}(\overline{G})$ with at least two vertices and at most one external vertex the following inequality holds,*

$$(13) \quad (d-2)N_g < d(|V| - 1),$$

where g is the induced subgraph of V in \overline{G} . I.e. g has all the edges of \overline{G} that have both vertices in V . Moreover, N_g is the sum of edge weights in g .

FIGURE 4. Appending an edge to the vertex z in G gives G_1 .

In Example 4.2 we only have the case $|V| = 2$ and $N_g = 1$, so that $f_G^{(\lambda)}(z)$ exists in any dimension greater than two.

4.3. General properties of graphical functions. Graphical functions should be considered as functions on $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. They have the following general properties:

Theorem 4.4. *Let G be a graph such that the graphical function $f_G^{(\lambda)}$ exists.*

(G1)

$$(14) \quad f_G^{(\lambda)}(z) = f_G^{(\lambda)}(\bar{z}).$$

(G2) $f_G^{(\lambda)}$ is a single-valued real analytic function on $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$.

(G3) The radius of convergence of the real analytic expansion $f_G^{(\lambda)}$ at $z_0 \in \overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ is the distance from z_0 to the nearest singularity of $f_G^{(\lambda)}$.

Let $\nu_z^>$ ($\nu_z^<$) be the sum of positive (negative) weights of edges adjacent to z . Let the dimension $d = 2\lambda + 2$ be even. Then, if $z_0 \in \{0, 1\}$, we have for $|z - z_0| < 1$ and coefficients $c_{\ell, m, n}(z_0) \in \mathbb{C}$:

$$(15) \quad f_G^{(\lambda)}(z) = \sum_{\ell=0}^{|V^{\text{int}}|} \sum_{m=M_{z_0}}^{\infty} \sum_{n=N_{z_0}}^{\infty} c_{\ell, m, n}(z_0) \log^{\ell}[(z - z_0)(\bar{z} - \bar{z}_0)](z - z_0)^m (\bar{z} - \bar{z}_0)^n,$$

where

$$M_{z_0}, N_{z_0} > -\lambda \nu_z^>.$$

If $z_0 = \infty$ we have for $|z| > 1$ and coefficients $c_{\ell, m, n}(\infty) \in \mathbb{C}$:

$$(16) \quad f_G^{(\lambda)}(z) = \sum_{\ell=0}^{|V^{\text{int}}|} \sum_{m=-\infty}^{M_{\infty}} \sum_{n=-\infty}^{N_{\infty}} c_{\ell, m, n}(\infty) \log^{\ell}(z\bar{z}) z^m \bar{z}^n,$$

where

$$M_{\infty}, N_{\infty} < -\lambda \nu_z^<.$$

Property (G1) is immediate by symmetry. It also follows from the parametric representation of graphical functions [25]. Property (G2) is proved in [25]. The proof of (G3) will be in [43]. There also exists an exact formula for (the maximal or minimal, resp.) M_{z_0} and N_{z_0} (see Conj. 4.12). In the case of odd dimensions there exist expansions at $0, 1, \infty$ which are similar to (15) and (16) with additional square roots [43].

4.4. Appending edges. Edges between external vertices give factors in a graphical function. In the case that an edge is appended to the vertex z creating a new vertex z (see Figure 4) the graphical functions are related by a differential equation [42, 24].

Lemma 4.5. *In the setup of Figure 4 we have*

$$(17) \quad \left(-\frac{1}{(z - \bar{z})} \partial_z \partial_{\bar{z}} (z - \bar{z}) + \frac{\lambda - 1}{z - \bar{z}} (\partial_z - \partial_{\bar{z}}) \right) f_{G_1}^{(\lambda)}(z) = \frac{1}{\Gamma(\lambda)} f_G^{(\lambda)}(z),$$

where $\Gamma(\lambda) = \int_0^\infty x^{\lambda-1} \exp(-x) dx$ is the gamma function.

The differential equation is particularly simple in $d = 4$ dimensions ($\lambda = 1$). In this case we (uniquely) obtain the graphical function of G_1 by single-valued integration with respect to z and \bar{z} (see [42]).

Lemma 4.6. *Let $\lambda = 1$. The differential operator on the left hand side of (17) has trivial kernel in the space of functions with general properties (G1) – (G3).*

Proof. Assume f with properties (G1) – (G3) is in the kernel of the differential operator. Because of (G2) $g(z) = \partial_z (z - \bar{z}) f(z)$ is meromorphic. We use (15) with $N_0, M_0, N_1, M_1 \geq 0$ (because $\nu_z^> = 1$ in G_1) and conclude that g is holomorphic on \mathbb{C} . From (16) with $N_\infty, M_\infty \leq -1$ we get $g(\infty) = 0$, hence, by Liouville's theorem, $g = 0$. Therefore $f = h(\bar{z})/(z - \bar{z})$ for some anti-holomorphic function h . With (G1) we obtain $h = 0$. \square

Beginning with the empty graphical function we can construct many graphical functions by appending edges (see [42, 15]).

Theorem 4.7. *Let G be a graph with external vertex width four (constructible in [42]), i.e. G can be constructed from the empty graph by adding edges between external vertices, permuting external vertices, and appending edges. If G has no edges between external vertices then we obtain in $d = 4$ dimensions,*

$$(18) \quad f_G^{(1)}(z) = P(z)/(z - \bar{z}),$$

where P is a single-valued multiple polylogarithm [7] of weight $2V^{\text{int}}(G)$.

The proof of the theorem will be in [43].

In $d = 4 - \epsilon$ dimensions edges can still be appended if one expands $f_G^{(\lambda)}$ into a Laurant series in ϵ to a given order. The procedure, however, is more subtle, see Sect. 4.8.

4.5. Identities. There exist many identities between graphical functions. A Fourier-identity relating planar duals was proved in [25]. Like in the case of periods there also exists a twist identity. Some complicated graphical functions can be calculated by a Gegenbauer technique [42, 43]. All known identities are included in a Maple package that (among other things) can calculate many graphical functions [46]. If a graphical function is inaccessible to all of these methods then sometimes it can still be calculated by parametric integration, due to F. Brown [8, 9] and E. Panzer [34, 35].

4.6. From graphical functions to periods. There exist several options to derive Feynman periods from graphical functions.

Firstly, one can specify the variable z in $f_G^{(\lambda)}(z)$ to 0, 1, or ∞ . In the case of the zig-zag graphs depicted in Figure 5 this method leads to a proof of a conjecture by D. Broadhurst and D. Kreimer in 1995 [5].

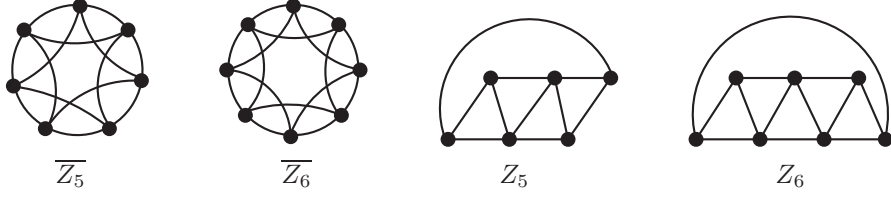


FIGURE 5. Completed (\overline{Z}_\bullet) and uncompleted (Z_\bullet) zig-zag graphs with five and six loops.

Theorem 4.8 (F. Brown, O. Schnetz, [15]). *The period of the graph Z_n is given by*

$$(19) \quad P(Z_n) = 4 \frac{(2n-2)!}{n!(n-1)!} \left(1 - \frac{1 - (-1)^n}{2^{2n-3}}\right) \zeta(2n-3).$$

Secondly, in integer dimensions $d \geq 3$ one can integrate $f_G^{(\lambda)}(z)$ over the external variable z [42]. This effectively makes z an internal variable. For a general graph in integer dimensions this is the best method to calculate Feynman periods. It is used in [46].

Thirdly, in the case of dimensional regularization we need to treat the dimension d as a parameter and expand periods in $4 - \epsilon$ ‘dimensions’ at $\epsilon = 0$. Using the second method one obtains an integration measure of $(z - \bar{z})^{2-\epsilon}$ which does not expand in ϵ into GSVHs (see Sect. 4.7). One has to resort to the following procedure to integrate over z : Add an edge of weight -1 between the external vertices 0 and z . Then, append an edge to the vertex z creating a new vertex z . Set $z = 0$, so that the newly appended edge cancels the previously added edge of weight -1 . This integrates over z in any possibly non-integer ‘dimension’ d . This method is used for perturbative calculations in dimensionally regularized ϕ^4 theory (see Sect. 4.9).

4.7. Generalized single-valued hyperlogarithms. Many graphical functions can be expressed in terms of iterated integrals [19]. Although, by (G2) of Sect. 4.4, graphical functions have only singularities at 0 , 1 , and ∞ it turns out that single-valued multiple polylogarithms [7] (i.e. letters 0 and 1 in the iterated integrals) are too restrictive. Let us consider the following example (which shows the same mechanism although it is not a graphical function).

Example 4.9. let $r > 0$. Define

$$(20) \quad f(z) = \frac{\log(z\bar{z}/r)}{z - r/\bar{z}}.$$

Then f is real analytic on $\overline{\mathbb{C}} \setminus \{0, \infty\}$ because the zero locus $z = r/\bar{z}$ in the denominator is canceled by the numerator. By general principles, there should exist a single-valued primitive of f which also is real analytic on $\overline{\mathbb{C}} \setminus \{0, \infty\}$. By single-valuedness the primitive is determined up to a rational function in \bar{z} . It is unique in the space of hyperlogarithms which vanish at 0 .

We generalize the above example in the sense that we allow the denominator to vanish at $z = (a\bar{z} + b)/(c\bar{z} + d)$ for some constants a, b, c, d if the zero is canceled by a zero in the numerator. A weight three example of this type is the single-valued primitive of $D(z)/(z - \bar{z})$ where D is the Bloch-Wigner dilogarithm (12) (see also [18]). Let \mathcal{G} be the space of these generalized single-valued hyperlogarithms (GSVHs). Upon differentiation with respect to z and \bar{z} we obtain the spaces $\partial_z \mathcal{G}$, $\partial_{\bar{z}} \mathcal{G}$, $\partial_{\bar{z}} \partial_z \mathcal{G}$. For

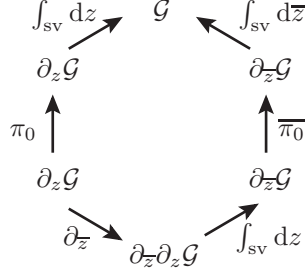


FIGURE 6. The inductive construction of GSVHs by a commutative hexagon.

example, f in (20) is in $\partial_z \mathcal{G}$. We would like to construct an algorithm for single-valued integration in $\partial_z \mathcal{G}$.

Functions in $\partial_z \mathcal{G}$ have expansions (15) and (16) for general $z_0 \in \overline{\mathbb{C}}$. The holomorphic residue res_{z_0} is the coefficient $c_{0,-1,0}(z_0)$, whereas the anti-holomorphic residue $\overline{\text{res}}_{z_0}$ is the coefficient $c_{0,0,-1}(z_0)$. let π_0 ($\overline{\pi}_0$) be the projection onto the (anti-)residue free part,

$$\begin{aligned} \pi_0 : \partial_z \mathcal{G} &\rightarrow \partial_z \mathcal{G} \quad , \quad f(z) \mapsto f(z) - \sum_{z_0 \in \mathbb{C}} \frac{\text{res}_{z_0}(f)}{z - z_0}, \\ \overline{\pi}_0 : \partial_{\bar{z}} \mathcal{G} &\rightarrow \partial_{\bar{z}} \mathcal{G} \quad , \quad f(z) \mapsto f(z) - \sum_{z_0 \in \mathbb{C}} \frac{\overline{\text{res}}_{z_0}(f)}{\bar{z} - \bar{z}_0}. \end{aligned}$$

The method to obtain single-valued primitives relies on the commutative hexagon in Figure 6, where \int_{sv} stands for single-valued integration.

Theorem 4.10. *The diagram in Figure 6 commutes.*

The proof of the theorem will be in [44]. By virtue of Figure 6 we can express a single-valued primitive with respect to z also as a single-valued primitive with respect to \bar{z} . Because single-valued integration in z (resp. \bar{z}) equals ordinary integration up to an anti-holomorphic (resp. holomorphic) function, knowing both integrands determines the single-valued primitive up to a constant (which is fixed by the condition that the single-valued primitive vanishes at $z = 0$). Using integration by parts at the bottom right arrow in Figure 6, we can reduce single-valued integration to lower weights. At weight zero a function in $\partial_z \mathcal{G}$ is a rational function with poles of the type $(z - c)^{-1}$, for some $c \in \mathbb{C}$. Therefore, at weight one the space GSVHs equals the space of ordinary single-valued hyperlogarithms. It is spanned by logarithms $\log[(z - c)(\bar{z} - \bar{c})]$.

Example 4.11 (Example 4.9 continued). In terms of iterated integrals (writing from right to left) we obtain (note that f is residue-free)

$$\log(z\bar{z}/r) = I(z, 0, 0) + I(\bar{z}, 0, 0) - I(r, 0, 0).$$

The single-valued primitive of f has the form

$$(21) \quad \int_{\text{sv}} f(z) dz = I(z, r/\bar{z}, 0, 0) + I(z, r/\bar{z}, 0)[I(\bar{z}, 0, 0) - I(r, 0, 0)] + g(\bar{z})$$

for some anti-holomorphic g . Differentiation with respect to \bar{z} yields

$$\partial_{\bar{z}}f(z) = \frac{1}{z\bar{z} - r} - r \frac{\log(z\bar{z}/r)}{(z\bar{z} - r)^2}.$$

Using integration by parts on the second term we find

$$\int_{\text{sv}} -r \frac{\log(z\bar{z}/r)}{(z\bar{z} - r)^2} dz = \frac{r \log(z\bar{z}/r)}{\bar{z}(z\bar{z} - r)} - \int_{\text{sv}} \frac{r}{z\bar{z}(z\bar{z} - r)} dz.$$

Adding the two terms which still have to be integrated, the factor $(z\bar{z} - r)$ cancels (as promised) and we obtain

$$\int_{\text{sv}} \partial_{\bar{z}}f(z) dz = \frac{r \log(z\bar{z}/r)}{\bar{z}(z\bar{z} - r)} + \frac{\log(z\bar{z})}{\bar{z}}.$$

In fact, there is an ambiguity in form of a rational function in \bar{z} . However, because the result has to be in $\partial_{\bar{z}}\mathcal{G}$, the ambiguity can only have simple poles and it is removed by the projection $\overline{\pi}_0$. The above expression has an anti-residue at $z = 0$ with value $\log(r)$. Subtraction yields

$$\overline{\pi}_0 \int_{\text{sv}} \partial_{\bar{z}}f(z) dz = \frac{\log(z\bar{z}/r)}{\bar{z} - r/z}.$$

Using the commutative hexagon we obtain by integration with respect to \bar{z} ,

$$(22) \quad \int_{\text{sv}} f(z) dz = I(\bar{z}, r/z, 0, 0) + I(\bar{z}, r/z, 0) [I(z, 0, 0) - I(r, 0, 0)] + h(z)$$

for some holomorphic function h . If we write (22) as hyperlogarithms in z whose coefficients are hyperlogarithms in \bar{z} we get (21) with $h(z)$ instead of $g(\bar{z})$. (Alternatively we may treat z and \bar{z} as independent variables in (22) and consider the limit $z \rightarrow 0$.) We conclude that in this example $h(z) = g(\bar{z})$ is a constant. This constant is zero because the single-valued integral is required to vanish at $z = 0$.

At four dimensions some (few) graphical functions exist which can be expressed in terms of ordinary single-valued multiple polylogarithms (see e.g. [15]). By far the most graphical functions which can be expressed in terms of iterated integrals are GSVHs which are not single-valued multiple polylogarithms. In $4 - \epsilon$ ‘dimensions’ every non-trivial graphical functions expands in ϵ with coefficients which are not single-valued multiple polylogarithms. Again, many such coefficients are GSVHs.

4.8. $4 - \epsilon$ dimensions. We can use the parametric representation of graphical functions [25] to define graphical function for non-integer d . The general properties (G1) and (G2) in Theorem 4.4 are still valid. In general, graphical functions in non-integer dimensions are complicated. However, for $d = 4 - \epsilon$ it is possible to expand these graphical functions into Laurant series in ϵ . For the coefficients (G3) holds, and often they can be expressed in terms of GSVHs.

The main tool for constructing these coefficients is again appending edges (see Sect. 4.4). Equation (17) can be solved iteratively by powers in ϵ . However, in this approach we cannot directly use Lemma 4.6 to avoid the kernel of the differential operator. We first need to subtract poles at $z = z_0$, $z_0 \in \{0, 1\}$, which are of order four (or higher) in $|z - z_0|$. For these singular contributions we need exact results; it is not sufficient to know them to a limited order in ϵ . In a renormalizable quantum field theory we only have to deal with ‘logarithmic’ singularities. I.e. the case of poles of order four suffices. This leading order asymptotic behavior of graphical functions is obtained by the following result which we (by now) leave as a very well tested conjecture.

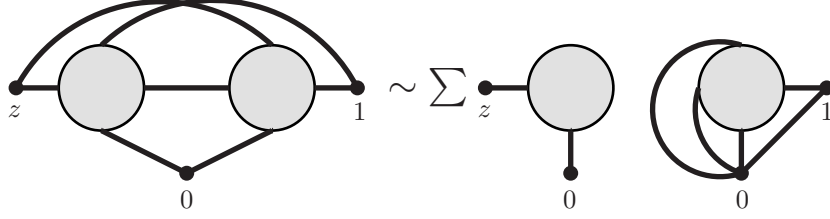


FIGURE 7. The asymptotic expansion of graphical functions at $z = 0$. The bold lines stand for sets of edges.

Conjecture 4.12. Let G be a graph with \mathcal{V}^{int} internal and $\mathcal{V}^{\text{ext}} = \{0, 1, z\}$ external vertices such that the graphical function $f_G^{(\lambda)}$ exists. Let $G[V]$ be the subgraph of G which is induced by V , i.e. the subgraph which contains the vertices V and all edges of G with both vertices in V . Further let $z_0 \in \{0, 1\}$ and $G[V = z_0]$ be the graph $G \setminus G[V]$ where one identifies all vertices in V with the vertex z_0 . Then (see Figure 7) we obtain asymptotic expansions at $z = z_0$ by

$$(23) \quad f_G^{(\lambda)}(z) = \sum_{V \subseteq \mathcal{V}^{\text{int}}} f_{G[V \cup \{z_0, z\}]}^{(\lambda)}(z) f_{G[V \cup \{z_0, z\} = z_0]}^{(\lambda)}(1 + O(|z - z_0|^2)),$$

and an asymptotic expansion at $z = \infty$ by

$$(24) \quad f_G^{(\lambda)}(z) = \sum_{V \subseteq \mathcal{V}^{\text{int}}} f_{G[V \cup \{0, 1\}]}^{(\lambda)} f_{G[V \cup \{0, 1\} = 0]}^{(\lambda)}(z) (1 + O(|z|^{-2})).$$

Note that on the right hand side of the above equations one has graphs with two external vertices (see (2)). The calculation of their functions amounts to calculating periods (see Sect. 3) which is much simpler than the calculation of a graphical function. Equations (23) and (24) are formally obtained by rescaling some internal variables $x_i \mapsto x_i|z|$ followed by a naive expansion in the integrand. The sum is over all possible ways to do this.

On top of appending edges there exists a variety of tools that allows one to calculate the ϵ -expansions of graphical functions in $4 - \epsilon$ dimensions to low orders in ϵ [45].

4.9. β , γ , and self-energy in dimensionally regularized ϕ^4 theory. Most efficiently one calculates ϕ^4 renormalization functions in the minimal subtraction scheme of dimensional regularization [28]. Presently it is possible to calculate the β function and the self-energy Σ to six loops, and the anomalous dimension γ to seven loops:

$$\begin{aligned}
 \beta = & \left(-\frac{18841427}{11520} - \frac{779603}{240}\zeta(3) + \frac{5663}{480}\pi^4 - \frac{63723}{10}\zeta(5) + \frac{6691}{1890}\pi^6 - \frac{8678}{5}\zeta(3)^2 \right. \\
 & + \frac{9}{5}\pi^4\zeta(3) - \frac{63627}{5}\zeta(7) + \frac{764621}{2304} + \frac{7965}{16}\zeta(3) - \frac{1189}{720}\pi^4 \\
 & + \frac{88181}{78750}\pi^8 - 4704\zeta(3)\zeta(5) - \frac{51984}{25}\zeta(5, 3) - 768\zeta(3)^3 - \frac{46112}{3}\zeta(9) \Big) g^7 \\
 & + \left(987\zeta(5) - \frac{5}{14}\pi^6 + 45\zeta(3)^2 + 1323\zeta(7) \right) g^6 \\
 (25) \quad & + \left(-\frac{3499}{48} - 78\zeta(3) + \frac{1}{5}\pi^4 - 120\zeta(5) \right) g^5 + \left(\frac{145}{8} + 12\zeta(3) \right) g^4 - \frac{17}{3}g^3 + 3g^2 \\
 \approx & -34776g^7 + 2848.5g^6 - 271.60g^5 + 32.549g^4 - 5.6666g^3 + 3g^2.
 \end{aligned}$$

This confirms (up to a trivial error in their $\zeta(9)$ coefficient) a recent result by M. Kompaniets and E. Panzer [30].

$$\begin{aligned}
\gamma &= \left(-\frac{214519}{5120} - \frac{52883}{1920}\zeta(3) - \frac{4247}{23040}\pi^4 + \frac{8023}{320}\zeta(5) - \frac{71}{1080}\pi^6 - \frac{523}{40}\zeta(3)^2 \right. \\
&\quad - \frac{1}{20}\pi^4\zeta(3) + \frac{3573}{40}\zeta(7) - \frac{2063}{210000}\pi^8 + 27\zeta(3)\zeta(5) + \frac{162}{25}\zeta(5,3) \Big) g^7 \\
&\quad + \left(\frac{73667}{9216} + \frac{295}{192}\zeta(3) + \frac{73}{1920}\pi^4 - \frac{37}{8}\zeta(5) + \frac{5}{756}\pi^6 - \frac{1}{2}\zeta(3)^2 \right) g^6 \\
&\quad + \left(-\frac{3709}{2304} + \frac{3}{16}\zeta(3) - \frac{1}{180}\pi^4 \right) g^5 + \frac{65}{192}g^4 - \frac{1}{16}g^3 + \frac{1}{12}g^2 \\
(26) \quad &\approx -124.15g^7 + 14.383g^6 - 1.9255g^5 + 0.33854g^4 - 0.0625g^3 + 0.083333g^2.
\end{aligned}$$

This confirms (and goes beyond) a recent six loop calculation of D.V. Batkovich, K.G. Chetyrkin, and M.V. Kompaniets [3].
















$$\begin{aligned}
\frac{\Sigma(p)}{p^2} &= \left[-\frac{27}{2}L^5 - \frac{3643}{24}L^4 + \left(-\frac{648011}{864} - 16\zeta(3) \right) L^3 \right. \\
&\quad + \left(-\frac{291187}{144} - 82\zeta(3) - \frac{1}{20}\pi^4 - 20\zeta(5) \right) L^2 \\
&\quad + \left(-\frac{1699885}{576} - \frac{32953}{192}\zeta(3) - \frac{11}{48}\pi^4 - \frac{211}{4}\zeta(5) - \frac{5}{378}\pi^6 + \zeta(3)^2 \right) L \\
&\quad - \frac{33992153}{18432} - \frac{683389}{4608}\zeta(3) - \frac{18403}{69120}\pi^4 - \frac{8681}{192}\zeta(5) \\
&\quad \left. - \frac{359}{18144}\pi^6 - \frac{83}{48}\zeta(3)^2 + \frac{1}{360}\pi^4\zeta(3) + \frac{5}{6}\zeta(7) \right] g^6 \\
&\quad + \left[\frac{9}{2}L^4 + \frac{1375}{36}L^3 + \left(\frac{12935}{96} + 2\zeta(3) \right) L^2 + \left(\frac{8353}{36} + 5\zeta(3) + \frac{1}{90}\pi^4 \right) L \right. \\
&\quad \left. + \frac{1874629}{11520} + \frac{6319}{1440}\zeta(3) + \frac{251}{14400}\pi^4 - \frac{1}{5}\zeta(5) \right] g^5 \\
&\quad + \left[-\frac{3}{2}L^3 - \frac{53}{6}L^2 - \frac{1867}{96}L - \frac{2017}{128} + \frac{3}{32}\zeta(3) \right] g^4 \\
(27) \quad &+ \left[\frac{1}{2}L^2 + \frac{7}{4}L + \frac{167}{96} \right] g^3 + \left[-\frac{1}{6}L - \frac{13}{48} \right] g^2,
\end{aligned}$$

where

















$$L = \frac{1}{2} \log \left(\frac{4\pi\Lambda^2}{\exp(C)p^2} \right)$$

with the renormalization scale Λ and Euler-Mascheroni constant $C = 0.577\dots$. This result confirms an unpublished five loop result by D. Broadhurst for the propagator $1/(p^2 - \Sigma(p))$ at $L = 0$ [4].

Currently a seven loop calculation of the ϕ^4 β function is running. A seven loop self-energy and an eight loop anomalous dimension are also within the range of this technology.

name	graph	numerical value	Aut	index	anc.	$-c_2$	remarks, [Lit]
weight		exact value					
P_1		1	48	—	P_1	—	$C_{1,1}^3$
0		1					
P_3		7.212 341 418	120	6	P_3	1	$C_{1,2}^5$, K_5 , [20]
3		$6Q_3$					
P_4		20.738 555 102	48	40	P_3	1	$C_{1,2}^6$, O_3 , [21]
5		$20Q_5$					
P_5		55.585 253 915	14	882	P_3	1	$C_{1,2}^7$, $\overline{C_7}$, [29]
7		$\frac{441}{8}Q_7$					
$P_{6,1}$		168.337 409 994	16	24192	P_3	1	$C_{1,2}^8$ [47]
9		$168Q_9$					
$P_{6,2}$		132.243 533 110	4	16	P_3	1	[5]
9		$\frac{1063}{9}Q_9 + 8Q_3^3$					
$P_{6,3}$		107.711 024 841	16	72	P_3^2	0	[5]
8		$256Q_8 + 72Q_3Q_5$					
$P_{6,4}$		71.506 081 796	1152	1728	$P_{6,4}$	0	$C_{1,3}^8$ [5], [38]
8		$-4096Q_8 + 288Q_3Q_5$					
$P_{7,1}$		527.745 051 766	18	405108	P_3	1	$C_{1,2}^9$ [5]
11		$\frac{33759}{64}Q_{11,1}$					
$P_{7,2}$		380.887 829 534	2	20	P_3	1	[5]
11		$\frac{62957}{192}Q_{11,1} + 9Q_{11,2} + 35Q_3^2Q_5$					
$P_{7,3}$		336.067 072 110	2	16	P_3	1	[5]
11		$\frac{73133}{240}Q_{11,1} + \frac{144}{5}Q_{11,2} + 20Q_3^2Q_5$					
$P_{7,4}$		294.035 314 185	4	320	P_3^2	0	[5]
10		$420Q_3Q_7 - 200Q_5^2$					
$P_{7,5}$		254.763 009 595	8	144	P_3^2	0	[5]
10		$-189Q_3Q_7 + 450Q_5^2$					
$P_{7,6}$		273.482 574 258	2	20	P_3	1	[5]
11		$\frac{14279}{64}Q_{11,1} - 51Q_{11,2} + 35Q_3^2Q_5$					
$P_{7,7}$		294.035 314 185	8	320	P_3^2	0	Fourier, twist
10		$P_{7,4}$					

name	graph	numerical value	Aut	index	anc.	$-c_2$	remarks, [Lit]
weight		exact value					
$P_{7,8}$		183.032 420 030	16	16	$P_{7,8}$	z_2	
11		$\frac{22383}{20}Q_{11,1} - \frac{4572}{5}Q_{11,2} + 1792Q_3Q_8 - 700Q_3^2Q_5$					
$P_{7,9}$		216.919 375 587	12	3	$P_{7,9}$	z_2	[6]
11		$\frac{92943}{160}Q_{11,1} - \frac{3381}{20}Q_{11,2} + 896Q_3Q_8 - \frac{1155}{4}Q_3^2Q_5$					
$P_{7,10}$		254.763 009 595	72	144	$P_{7,10}$	0	$K_3 \square K_3$, Fourier
10		$P_{7,5}$					
$P_{7,11}$		200.357 566 429	18	?	$P_{7,11}$	z_3	$C_{1,3}^9$, [34]
11?		$Q_{11,3}$					
$P_{8,1}$		1 716.210 576 104	20	2635776	P_3	1	$C_{1,2}^{10}$
13		$1716Q_{13,1}$					
$P_{8,2}$		1 145.592 929 599	2	12	P_3	1	
13		$\frac{25147347}{22400}Q_{13,1} - \frac{16881}{1400}Q_{13,2} + \frac{459}{112}Q_{13,3} + \frac{1305}{8}Q_3^2Q_7 - 135Q_3Q_5^2$					
$P_{8,3}$		1 105.107 697 390	4	1280	P_3	1	
13		$298Q_{13,1} + 56Q_{13,2} - 20Q_{13,3} - 280Q_3^2Q_7 + 800Q_3Q_5^2$					
$P_{8,4}$		966.830 801 986	1	12	P_3	1	
13		$\frac{17124243}{22400}Q_{13,1} - \frac{19689}{1400}Q_{13,2} + \frac{1755}{112}Q_{13,3} + \frac{9}{8}Q_3^2Q_7 + 135Q_3Q_5^2$					
$P_{8,5}$		844.512 518 603	4	24	P_3^2	0	
12		$1536Q_{12,1} - 1280Q_{12,2} + 36Q_3Q_9 + \frac{1299}{2}Q_5Q_7$					
$P_{8,6}$		904.280 824 357	4	32	P_3	1	
13		$\frac{214841}{336}Q_{13,1} - \frac{423}{7}Q_{13,2} + \frac{705}{14}Q_{13,3} + 183Q_3^2Q_7$					
$P_{8,7}$		847.646 115 639	2	144	P_3	1	
13		$\frac{2061501}{2800}Q_{13,1} + \frac{13527}{175}Q_{13,2} - \frac{675}{14}Q_{13,3}$					
$P_{8,8}$		847.646 115 639	2	144	P_3	1	twist
13		$P_{8,7}$					
$P_{8,9}$		904.280 824 357	2	32	P_3	1	twist
13		$P_{8,6}$					
$P_{8,10}$		735.764 103 468	2	72	P_3^2	0	
12		$1536Q_{12,1} - 1280Q_{12,2} - \frac{63}{2}Q_3Q_9 + \frac{2493}{4}Q_5Q_7$					
$P_{8,11}$		805.347 388 507	4	16	P_3^2	0	
12		$\frac{10240}{69}Q_{12,1} + \frac{81920}{69}Q_{12,2} - \frac{2560}{69}Q_{12,3} + \frac{45503}{69}Q_3Q_9 + \frac{305}{46}Q_5Q_7 - 12Q_3^4$					

name	graph	numerical value	Aut	index	anc.	$-c_2$	remarks, [Lit]
weight		exact value					
$P_{8,12}$		688.898 361 296	2	288	P_3^2	0	
12		$1024Q_{12,2} - 1008Q_3Q_9 + 1800Q_5Q_7$					
$P_{8,13}$		742.977 090 366	1	4	P_3	1	
13		$\frac{10087273}{9600}Q_{13,1} + \frac{8007}{200}Q_{13,2} - \frac{813}{16}Q_{13,3} + \frac{2247}{8}Q_3^2Q_7 - 465Q_3Q_5^2$					
$P_{8,14}$		749.818 622 995	1	4	P_3	1	
13		$\frac{41038969}{67200}Q_{13,1} - \frac{30129}{1400}Q_{13,2} + \frac{1611}{112}Q_{13,3} + \frac{153}{8}Q_3^2Q_7 + 105Q_3Q_5^2$					
$P_{8,15}$		805.347 388 507	2	16	P_3^2	0	twist
12		$P_{8,11}$					
$P_{8,16}$		633.438 914 549	32	576	P_3^3	0	$[-10080Q_5^2]$
11,10		$-\frac{31851}{5}Q_{11,1} + \frac{24336}{5}Q_{11,2} - 10240Q_3Q_8 + 5040Q_3^2Q_5 - 8192Q_{10} + 9648Q_3Q_7$					
$P_{8,17}$		589.354 510 434	2	8	P_3	1	$[-1410Q_3Q_5^2]$
13		$\frac{15548993}{4800}Q_{13,1} - \frac{17313}{100}Q_{13,2} + \frac{267}{8}Q_{13,3} + 512Q_3Q_{10} + 2304Q_5Q_8 - \frac{825}{4}Q_3^2Q_7$					
$P_{8,18}$		641.723 358 297	2	48	P_3^2	0	
12		$727Q_3Q_9 - \frac{735}{2}Q_5Q_7 + 72Q_3^4$					
$P_{8,19}$		598.617 690 750	4	32	P_3^2	0	
12		$\frac{10240}{69}Q_{12,1} + \frac{81920}{69}Q_{12,2} - \frac{2560}{69}Q_{12,3} + \frac{13970}{69}Q_3Q_9 + \frac{11020}{23}Q_5Q_7 - 84Q_3^4$					
$P_{8,20}$		641.346 699 620	1	6	P_3	1	$[+\frac{1035}{2}Q_3Q_5^2]$
13		$\frac{4375463}{44800}Q_{13,1} + \frac{383001}{2800}Q_{13,2} - \frac{23607}{224}Q_{13,3} - 256Q_3Q_{10} + 256Q_5Q_8 - \frac{1953}{16}Q_3^2Q_7$					
$P_{8,21}$		742.977 090 366	2	4	P_3	1	Fourier, twist
13		$P_{8,13}$					
$P_{8,22}$		735.764 103 468	4	72	P_3^2	0	twist
12		$P_{8,10}$					
$P_{8,23}$		589.354 510 434	2	8	P_3	1	twist
13		$P_{8,17}$					
$P_{8,24}$		414.873 975 722	8	144	$P_{7,8}$	z_2	$[+\frac{17577}{2}Q_3^2Q_7 + 10800Q_3Q_5^2]$
13		$-\frac{40309047}{1400}Q_{13,1} - \frac{353601}{350}Q_{13,2} + \frac{48051}{28}Q_{13,3} - 17920Q_3Q_{10} - 19840Q_5Q_8$					
$P_{8,25}$		641.723 358 297	4	48	P_3^2	0	Fourier, twist
12		$P_{8,18}$					
$P_{8,26}$		500.445 152 216	4	6	$P_{7,9}$	z_2	$[-\frac{1215}{2}Q_3Q_5^2]$
13		$\frac{25114323}{22400}Q_{13,1} - \frac{113979}{1400}Q_{13,2} + \frac{4443}{112}Q_{13,3} - 896Q_3Q_{10} + 1984Q_5Q_8 + \frac{1701}{8}Q_3^2Q_7$					
$P_{8,27}$		598.617 690 750	4	32	$P_{7,10}$	0	Fourier
12		$P_{8,19}$					















name	graph	numerical value	Aut	index	anc.	$-c_2$	remarks, [Lit]
weight		exact value					
$P_{8,28}$		500.445 152 216	4	6	$P_{7,9}$	z_2	twist
13		$P_{8,26}$					
$P_{8,29}$		553.273 794 612	2	1	$P_{7,9}$	z_2	
13		$\frac{78907643}{89600}Q_{13,1} - \frac{306689}{5600}Q_{13,2} + \frac{16987}{448}Q_{13,3} + \frac{10129}{32}Q_3^2Q_7 - \frac{2275}{4}Q_3Q_5^2$					
$P_{8,30}$		≈ 505.5	2	?	$P_{7,11}$	z_3	[36]
13		?					
$P_{8,31}$		460.088 538 246	4	8	$P_{7,8}$	z_2	$[-7330Q_3Q_5^2]$
13		$\frac{67363763}{5600}Q_{13,1} - \frac{36487}{175}Q_{13,2} - \frac{1913}{7}Q_{13,3} + 1792Q_3Q_{10} + 7936Q_5Q_8 + 98Q_3^2Q_7$					
$P_{8,32}$		470.720 125 534	16	17280	$P_{8,32}$	0	
12		$-\frac{81920}{23}Q_{12,1} - \frac{655360}{23}Q_{12,2} + \frac{20480}{23}Q_{12,3} + \frac{8760}{23}Q_3Q_9 + \frac{15660}{23}Q_5Q_7$					
$P_{8,33}$		468.038 498 992	2	?	$P_{8,33}$	z_3	
13		$Q_{13,4}$					
$P_{8,34}$		470.720 125 534	16	17280	$P_{8,34}$	0	twist
12		$P_{8,32}$					
$P_{8,35}$		≈ 460.2	16	?	$P_{8,35}$	z_2	[36]
13		$P_{8,31}?$					
$P_{8,36}$		≈ 505.5	10	?	$P_{8,36}$	z_3	[36]
13		$P_{8,30}?$					
$P_{8,37}$		≈ 422.9	2	?	$P_{8,37}$	$(3, 7)$	[36]
?		?					
$P_{8,38}$		≈ 386.6	4	?	$P_{8,38}$	$(4, 5)$	[36]
?		?					
$P_{8,39}$		≈ 384.2	8	?	$P_{8,39}$	$(3, 8)$	[36]
?		?					
$P_{8,40}$		≈ 312.1	320	?	$P_{8,40}$	z_4	$C_{1,4}^{10}$, [36]
13		?					
$P_{8,41}$		≈ 323.3	240	?	$P_{8,41}$	$(6, 3)$	$C_{1,3}^{10}$, [36]
?		?					

Table 3: The census of ϕ^4 periods. The numbers Q_\bullet are listed in Table 1. All known periods except for $P_{7,11}$ [34] can be calculated with [46]. See also [40] for explanations.

REFERENCES

- [1] Y. André, *Galois theory, motives and transcendental numbers*, In *Renormalization and Galois theories*, 15 of *IRMA Lect. Math. Theor. Phys.*, Eur. Math. Soc., Zürich, 165–177 (2009).
- [2] T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, *Revised value of the eighth-order QED contribution to the anomalous magnetic moment of the electron*, Phys. Rev. D **77** 053012 (2008).
- [3] D.V. Batkovich, K.G. Chetyrkin, M.V. Kompaniets, *Six loop analytical calculation of the field anomalous dimension and the critical exponent η in $O(n)$ -symmetric ϕ^4 model* arXiv:1601.01960 [hep-th] (2016).
- [4] D. Broadhurst, *Multi-loop calculations without subtractions: 5-loop propagator in ϕ^4 theory*, unpublished notes OUT-4102-46 (1993).
- [5] D. Broadhurst, D. Kreimer, *Knots and numbers in ϕ^4 theory to 7 loops and beyond*, Int. J. Mod. Phys. C **6**, 519 (1995).
- [6] D. Broadhurst, *Multiple zeta values and other periods in quantum field theory*, conference talk, Bristol, 4 May 2011.
- [7] F. Brown, *Single-valued multiple polylogarithms in one variable*, C.R. Acad. Sci. Paris, Ser. I **338**, 527–532 (2004).
- [8] F. Brown, *The massless higher-loop two-point function*, Commun. Math. Phys. **287** 925–958 (2009).
- [9] F. Brown, *On the periods of some Feynman integrals*, arXiv:0910.0114 [math.AG] (2009).
- [10] F. Brown, *Single-valued periods and multiple zeta values*, arXiv:1309.5309 [math.NT] (2013).
- [11] F. Brown, *Feynman amplitudes and cosmic Galois group*, arXiv:1512.06409 [math-ph] (2015).
- [12] F. Brown, *Notes on motivic periods*, arXiv:1512.06410 [math.NT] (2015).
- [13] F. Brown, D. Doryn, *Framings for graph hypersurfaces*, arXiv:1301.3056v1 [math.AG] (2013).
- [14] F. Brown, O. Schnetz, *A $K3$ in ϕ^4* , Duke Mathematical Journal, Vol. 161, No. 10 (2012).
- [15] F. Brown, O. Schnetz, *Single-valued multiple polylogarithms and a proof of the zig-zag conjecture*, Jour. of Numb. Theory **148**, 478–506 (2015).
- [16] F. Brown, O. Schnetz, *Modular forms in quantum field theory*, Commun. Numb. Theor. Phys. **7** (2013) 293–325.
- [17] F. Brown, O. Schnetz, K. Yeats, *Properties of c_2 invariants of Feynman graphs*, Adv. Theor. Math. Phys. **18**, no. 2, 323–362 (2014).
- [18] F. Chavez, C. Duhr, *Three-mass triangle integrals and single-valued polylogarithms*, arXiv:1209.2722 [hep-ph] (2012).
- [19] K. Chen, *Algebras of Iterated Path Integrals and Fundamental Groups*, Transactions of the American Mathematical Society, Vol. 156, 359–379 (1971).
- [20] K.G. Chetyrkin, A.L. Kataev, F.V. Tkachov, *New Approach to Evaluation of Multiloop Feynman Integrals: The Gegenbauer Polynomial x -Space Technique*, Nucl. Phys. B **174**, 345–377 (1980).
- [21] K.G. Chetyrkin, F.V. Tkachov, *Integration by Parts: The Algorithm to Calculate β -Functions in 4 Loops*, Nucl. Phys. B **192**, 159–204 (1981).
- [22] D. Doryn, *The c_2 invariant is invariant*, arXiv:1312.7271 [math.AG] (2013).
- [23] J. Drummond, C. Duhr, P. Heslop, J. Pennington, V. A. Smirnov, *Leading singularities and off-shell conformal integrals*, arXiv:1303.6909v2 [hep-th] (2013).
- [24] M. Golz, *Graphical functions in parametric space*, master thesis (2015).
- [25] M. Golz, E. Panzer, O. Schnetz, *Graphical functions in parametric space*, arXiv:1509.07296 [math-ph] (submitted to Letters in Math. Phys.) (2015).
- [26] A. B. Goncharov, *Galois symmetries of fundamental groupoids and noncommutative geometry*, Duke Math. J. Volume 128, no. 2, 209–284 (2005).
- [27] D. Hanneke, S. Fogwell, G. Gabrielse, *New Measurement of the Electron Magnetic Moment and the Fine Structure Constant*, Phys. Rev. Lett. **100** (2008) 120801.
- [28] J.C. Itzykson, J.B. Zuber, *Quantum Field Theory*. Mc-Graw-Hill, (1980).
- [29] D.I. Kazakov, *The method of uniqueness, a new powerful technique for multiloop calculations*, Phys. Lett. B **133** no. 6, 406 (1983).

- [30] M. Kompaniets, E. Panzer, *Six loop beta function in ϕ^4 model*, talk given by M. Kompaniets at 23rd July 2015.
- [31] M. Kontsevich, D. Zagier, *Periods*, in: B. Engquis and W. Schmid, editors, *Mathematics unlimited - 2001 and beyond*, Springer Verlag, Berlin, Heidelberg, New York 771 (2001).
- [32] S. Laporta, E. Remiddi, *The analytical value of the electron $(g-2)$ at order α^3 in QED*, Phys. Lett. B 379 (1996) 283.
- [33] N. Nakanishi, *Feynman-parametric formula for the Hankel-transformed position-space Feynman integral*, Progress of Theoretical Physics 42 966977 (1969).
- [34] E. Panzer, *Feynman integrals and hyperlogarithms*, PhD theses, Humboldt-Universität zu Berlin, arXiv:1506.07243 [math-ph] (2014).
- [35] E. Panzer, *Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals*, Computer Physics Communications 188 148-166 (2015).
- [36] E. Panzer, privat communication, May 3, 2016.
- [37] E. Panzer, O. Schnetz, *The Galois coaction on ϕ^4 periods*, arXiv:1603.04289 [hep-th] (submitted to Commun. Numb. Theor. Phys.) (2016).
- [38] O. Schnetz, *Calculation of the ϕ^4 6-loop non-zeta transcendental*, arXiv:hep-th/9912149 (1999).
- [39] O. Schnetz, *The geometry of one-loop amplitudes*, arXiv:1010.5334 (2010).
- [40] O. Schnetz, *Quantum periods: A census of ϕ^4 transcendentals*, Commun. Numb. Theor. Phys. 4 no. 1, 1-48 (2010).
- [41] O. Schnetz, *Quantum field theory over \mathbb{F}_q* , Electron. J. Comb. 18N1:P102 (2011).
- [42] O. Schnetz, *Graphical functions and single-valued multiple polylogarithms*, Commun. Numb. Theor. Phys. 8 (2014) 589-675.
- [43] O. Schnetz, *Graphical functions*, in preparation.
- [44] O. Schnetz, *Generalized single-valued hyperlogarithms*, in preparation.
- [45] O. Schnetz, *Seven loops ϕ^4* , in preparation.
- [46] O. Schnetz, *hyperlog-procedures*, in preparation.
- [47] N. I. Ussyukina, *Calculation of multiloop diagrams in high orders of perturbation theory*, Phys. Lett. B 267, 382 (1991).
- [48] K. Yeats, *A few c_2 invariants of circulant graphs*, arXiv:1507.06974 [math.CO] (2015).
- [49] K. Yeats, privat communication, June 7, 2016.
- [50] D. Zagier, *The dilogarithm function in geometry and number theory*, in: Number theory and related topics. Papers presented at the Ramanujan Colloquium, Bombay 1988, Studies in Mathematics no. 12, TIFR and Oxford University Press, pp. 231-249 (1989) and J. Math. Phys. Sci. no. 22 131-145 (1988).